

Charged rotating BTZ black holes in noncommutative space and torsion gravity

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Abstract

We consider charged rotating BTZ black holes in noncommutative space by use of Chern-Simons theory formulation of $2 + 1$ dimensional gravity. The noncommutativity between the radial and the angle variables is introduced through the Seiberg-Witten map for gauge fields, and the deformed geometry to the first order in the noncommutative parameter is calculated. It is found that the deformation also induces nontrivial torsion, and the framework of Einstein-Cartan theory appears to be suitable to investigate the equations of motion. Though the deformation is indeed nontrivial, the deformed and the original Einstein equations are found to be related by a rather simple coordinate change.

1 Introduction

It is widely believed that at a very high energy scale, such as Planck or the string scale, the notion of smooth geometry is to be altered due to the effect of quantum gravity; quantum fluctuation of spacetime itself becomes significant and may not be appropriate to treat it as a perturbation around a classical geometry. Though we have not yet known a suitable framework for such a *quantum geometry*, there are several proposals that may capture some expected features of it. The noncommutative geometry [1] is one of such proposals. Here, the space-time coordinates are no longer regarded as *c*-numbers but the ones obeying some quantum algebra. It usually introduces a length scale which has been expected to work as a natural UV cutoff. Quantum field theory formulated on a noncommutative geometry also exhibits various intriguing behavior such as the UV/IR mixing [2] and exhibits sting-like properties [3].

The rich structure that quantum field theory on noncommutative geometries may entice us to consider gravitational theory on noncommutative space. There have been several attempts in the past [4], but it is not easy to investigate concrete solutions due to complicated structures. Gravity in $2 + 1$ dimensions may be an exception. For instance, the Poisson brackets of $SL(2, \mathbf{R})$ were studied in [5] and families of deformation were found leading to a discrete spectrum for time operator. In the [6], an effective metric in noncommutative geometry was sourced by delocalized mass and charge due to the minimal length. In addition, $3D$ gravity is known to have a description in terms of Chern-Simons theory [7]. In this case, one may take advantage of the Seiberg-Witten map that relates a theory on commutative space to a corresponding theory on a noncommutative space. To mention a few examples: in the study [8], the Seiberg-Witten map was used to modify algebraic relation and found no first order correction as expected in the canonical treatment in the noncommutative geometry, as long as classical torsion is excluded. In the [9], the ambiguity in the metric due to gauge transformation were discussed and fixed by introducing nonminimal coupled scalar and nontrivial potential. In the [10], it was argued that cosmological constant gets quantized in the noncommutative Chern-Simons gravity.

On the other hand, the three-dimensional anti-de Sitter (AdS) space admits the black hole solution [11] and its charged counterpart [12]. In the [13], a constant gauge field was introduced in coupled with the Chern-Simon action and it resulted to a mixing of mass and angular momentum defined in the original BTZ. In the [14], a noncommutative deformation in polar coordinates was introduced via the Seiberg-Witten map and a noncommutative neutral BTZ black hole metric up to the first order in noncommutative parameter θ was obtained. However, this result appeared in conflict with that in [8] for its first order correction in metric. Before we could solve the puzzle, it is useful to review their construction.

In [14], a noncommutative deformation of a neutral rotating BTZ black hole solution is investigated based on a commutation relation in the polar coordinates, $[r^2, \varphi] = 2i\theta$.¹ The solution is written in terms of Chern-Simons gauge fields and the noncommutative deformation is introduced by the Seiberg-Witten map. The resultant metric, to the first order in θ , reads ²

$$ds^2 = -f^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 N^\phi dt d\varphi + \left(r^2 - \frac{\theta B}{2}\right) d\varphi^2 + \mathcal{O}(\theta^2), \quad (1.1)$$

¹We will review the noncommutativity deformation in more detail in Section 2.

²We have corrected the signs in f^2 and $d\varphi^2$ parts in [14].

with

$$N^\phi = -\frac{r_+ r_-}{\ell r^2}, \quad (1.2)$$

$$f^2 = \frac{r^2 - r_+^2 - r_-^2}{\ell^2} - \frac{\theta B}{2\ell^2}, \quad (1.3)$$

$$\hat{N}^2 = \frac{1}{\ell^2 r^2} \left[(r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2} (2r^2 - r_+^2 - r_-^2) \right], \quad (1.4)$$

where r_+ represents the horizon radius of the undeformed metric (the explicit forms are given in Appendix A). Introducing noncommutativity involves two extra $U(1)$ gauge fields $B_\mu^{(\pm)}$ which are chosen as $B_\varphi^{(\pm)} = B$ with a constant B . Some properties of this deformed black hole solution are investigated in [14]; such as the locations of various horizons.

We, however, confirm that this metric satisfies the vacuum Einstein equation to the first order, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 + \mathcal{O}(\theta^2)$, if the corrections of the metric in [14] are taken into account. This fact suggests that there should exist another coordinate system in which the metric looks like a pure AdS_3 , since in $2+1$ dimensions all the vacuum solutions with a negative cosmological constant are locally AdS_3 . It turns out that, by making a coordinate transformation

$$r \rightarrow \tilde{r} + \frac{\theta B}{4\tilde{r}}, \quad (1.5)$$

and taking the first order in θ , (1.1) comes back to the undeformed BTZ black hole solution; namely the first order corrections can be eliminated.³ The angular part of the metric is $\tilde{r}^2 d\varphi^2$, and then \tilde{r} is regarded as a standard radial coordinate. Consequently, there is no difference between the deformed BTZ black hole and the undeformed one. The change is only made in r , and r and \tilde{r} are asymptotically the same; the geometrical structure near the boundary would not be changed. Thus, various characteristic properties of black hole geometries, thermodynamic quantities such as the Hawking temperature or entropy, or the motion of particles, appear to be equivalent. This equivalence may be due to the fact that the vacuum solution of $2+1$ dimensional gravity with negative cosmological constant is essentially unique. This motivates us to investigate a class of solutions that are not vacuum solutions. Namely, in this paper, we are going to explore charged rotating BTZ black hole solutions in noncommutative space.

The organization of the paper is as follows: In Section 2, a noncommutative deformation is formulated by use of a Seiberg-Witten map in the Chern-Simons framework of $2+1$ dimensional gravity. We start with a charged rotating BTZ solution and obtain deformed gauge fields, vielbeins, and spin connections. In Section 3, we investigate the properties of noncommutative charged rotating BTZ black hole solutions. There appears nontrivial torsion and the deformed equations of motion are found to be nicely fitted in to the framework of Einstein-Cartan theory of torsion gravity. The relation between the deformed and the original solutions through a coordinate change is argued. We conclude the paper with discussion and overview in Section 4. Appendices are given to summarize our convention and to explain some technical details.

³There is a subtle issue about the regions covered by these coordinates, which will be discussed in Section 3.4.

2 Three dimensional gravity in noncommutative space

2.1 Noncommutativity in polar coordinates

A noncommutative space is introduced by applying the following commutation relations for coordinates,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}. \quad (2.1)$$

Since timelike noncommutativity is known to have several difficulties, such as acausality or nonunitarity [15], we restrict ourselves to considering a purely spatial noncommutativity; for example, $[x, y] = i\theta$ with θ being the parameter of noncommutativity, in $2 + 1$ dimensions. In this paper, we consider charged rotating BTZ black holes that are conveniently represented in terms of polar coordinates (t, r, φ) and we would introduce a noncommutativity between r and φ coordinates. As suggested in the [14], the noncommutative relation

$$[r^2, \varphi] = 2i\theta, \quad (2.2)$$

is a natural choice; this is because the standard spatial noncommutative relation $[x, y] = i\theta$ can be recovered by use of the polar coordinates and (2.2) to the first order in θ , as $[x, y] = [r \cos \varphi, r \sin \varphi] = i\theta + \mathcal{O}(\theta^2)$. We thus adopt this noncommutative relation and will consider a θ -deformed charged rotating BTZ black hole solution.

2.2 Charged rotating BTZ black hole solutions and Chern-Simons theory

2.2.1 Chern-Simons formulation of Einstein-Maxwell theory

We start with Einstein-Maxwell theory in $(2 + 1)$ dimensions,

$$I = I_{\text{gravity}} + I_{\text{gauge}}, \quad (2.3)$$

$$I_{\text{gravity}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} (R - 2\Lambda), \quad I_{\text{gauge}} = -\frac{1}{4\lambda^2} \int d^3x \sqrt{-g} f_{\mu\nu} f^{\mu\nu}, \quad (2.4)$$

where λ is the coupling constant of $U(1)$ gauge field a_μ whose field strength is $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. I_{gravity} part can be written as Chern-Simons action by use of two $SU(1, 1) \simeq SO(1, 2)$ connection 1-forms (relevant conventions are summarized in Appendix A),

$$A^{(\pm)a} = \omega^a \pm \frac{1}{\ell} e^a \quad (2.5)$$

as

$$S = I_{CS}[A^{(+)}] - I_{CS}[A^{(-)}], \quad (2.6)$$

$$I_{CS}[A] = \frac{k}{4\pi} \int \text{tr} \left[A dA + \frac{2}{3} A A A \right], \quad (2.7)$$

where the Chern-Simons level is identified with $k = -\frac{\ell}{4G}$.

For the gauge part, by use of the following definition of Hodge star, $*(dx^\mu \wedge dx^\nu) = \sqrt{|g|} \epsilon^{\mu\nu}{}_\rho dx^\rho$, it can be written as

$$I_{\text{gauge}} = -\frac{1}{4\lambda^2} \int d^3x \sqrt{-g} f_{\mu\nu} f_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} = -\frac{1}{2\lambda^2} \int f \wedge *f, \quad (2.8)$$

The equations of motion with respect to the variation of A_μ^a are

$$\frac{k}{4\pi}\epsilon^{\mu\rho\sigma}\left[\partial_\rho A_\sigma^{(\pm)a}-\frac{\epsilon^a{}_{bc}}{2}A_\rho^{(\pm)b}A_\sigma^{(\pm)c}\right]=\frac{\ell}{2}\cdot e\mathcal{T}^{\mu\rho}e_\rho^a. \quad (2.9)$$

where $\mathcal{T}^{\mu\rho}$ is the energy momentum tensor of $U(1)$ gauge field a_μ ,

$$\mathcal{T}_{\mu\nu}=\frac{1}{\lambda^2}\left[f_{\mu\rho}f_{\nu\sigma}g^{\rho\sigma}-\frac{1}{4}g_{\mu\nu}f^2\right]. \quad (2.10)$$

In terms of the vielbeins and the spin connections, the equations of motion can be represented as

$$\frac{k}{2\pi}\epsilon^{\mu\rho\sigma}\left[\partial_\rho\omega_\sigma^a-\frac{\epsilon^a{}_{bc}}{2}\omega_\rho^b\omega_\sigma^c-\frac{\epsilon^a{}_{bc}}{2\ell^2}e_\rho^be_\sigma^c\right]=\ell\cdot e\mathcal{T}^{\mu\rho}e_\rho^a, \quad (2.11)$$

$$\epsilon^{\mu\rho\sigma}\left(\partial_\rho e_\sigma^a-\epsilon^a{}_{bc}e_\rho^be_\sigma^c\right)=0. \quad (2.12)$$

The second equation is the torsion free condition, $T^a = De^a = de^a + \omega^a{}_be^b = 0$, while it is straight forward to see that the first one is the Einstein equation,

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-\frac{1}{\ell^2}g_{\mu\nu}=8\pi G\mathcal{T}_{\mu\nu}. \quad (2.13)$$

2.2.2 Charged rotating BTZ black hole

The charged rotating BTZ black hole solution is given in [12] as

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2\left(d\varphi-\frac{4GJ}{r^2}dt\right)^2 \quad (2.14)$$

$$=-h(r)dt^2+\frac{dr^2}{f(r)}+r^2d\varphi^2-\frac{2\gamma}{\ell}dtd\varphi, \quad (2.15)$$

$$f(r)=-8GM+\frac{r^2}{\ell^2}+\frac{16G^2J^2}{r^2}-8\pi GQ^2\ln r=\frac{1}{\ell^2}\left(-\alpha+r^2+\frac{\gamma^2}{r^2}-\beta\ln r\right), \quad (2.16)$$

$$f_{tr}=\frac{\lambda Q}{r}, \quad (2.17)$$

where Q is the electric charge of the black hole and we have defined

$$\alpha=8GM\ell^2, \quad \beta=8\pi G\ell^2Q^2, \quad \gamma=4GJ\ell, \quad (2.18)$$

and $h(r)=f(r)-\frac{\gamma^2}{\ell^2r^2}$.

From this metric, we can choose a set of convenient, but not unique, expressions of the vielbeins and the spin connections as follows,

$$e^0=\sqrt{h(r)}dt+\frac{\gamma}{\ell\sqrt{h(r)}}d\varphi, \quad e^1=\frac{1}{\sqrt{f(r)}}dr, \quad e^2=r\sqrt{\frac{f(r)}{h(r)}}d\varphi, \quad (2.19)$$

$$\omega^0=-\frac{\gamma h'(r)}{2\ell r\sqrt{h(r)}}dt-\sqrt{h(r)}d\varphi, \quad \omega^1=\frac{\gamma h'(r)}{2\ell r h(r)\sqrt{f(r)}}dr, \quad \omega^2=-\frac{h'(r)}{2}\sqrt{\frac{f(r)}{h(r)}}dt, \quad (2.20)$$

where the prime ' denotes the derivative with respect to r . The Chern-Simons gauge fields are defined by (2.5):

$$A^{(\pm)0} = \pm \frac{1}{\ell} \left(\sqrt{h(r)} \mp \frac{h'(r)}{2r} \frac{\gamma}{\sqrt{h(r)}} \right) dt - \left(\sqrt{h(r)} \mp \frac{1}{\ell^2} \frac{\gamma}{\sqrt{h(r)}} \right) d\varphi, \quad (2.21)$$

$$A^{(\pm)1} = \frac{1}{\ell \sqrt{f(r)}} \left(\frac{\gamma h'(r)}{2r h(r)} \pm 1 \right) dr, \quad (2.22)$$

$$A^{(\pm)2} = \frac{r}{\ell} \sqrt{\frac{f(r)}{h(r)}} \left(-\frac{\ell h'(r)}{2r} dt \pm d\varphi \right). \quad (2.23)$$

2.3 Noncommutative Chern-Simons theory

The Chern-Simons formulation of noncommutative three-dimensional gravity has been considered in [13, 16]. In Lorentzian version [16], it has been shown that the theory involves two extra $U(1)$ gauge fields $B_\mu^{(\pm)}$ that will be decoupled in the commutative limit, and the gauge group are $U(1, 1) \times U(1, 1)$ rather than $SO(1, 2) \times SO(1, 2)$ of the commutative case. The the action of noncommutative Chern-Simons theory is

$$\hat{I}_{CS}[\mathcal{A}^{(\pm)}] = \frac{k}{4\pi} \int \text{tr} \left[\mathcal{A}^{(\pm)} \star d\mathcal{A}^{(\pm)} + \frac{2}{3} \mathcal{A}^{(\pm)} \star \mathcal{A}^{(\pm)} \star \mathcal{A}^{(\pm)} \right], \quad (2.24)$$

where

$$f \star g = \frac{1}{i!j!} f_{\mu_1 \dots \mu_i} \star g_{\nu_1 \dots \nu_j} (dx^{\mu_1} \dots dx^{\mu_i}) \wedge (dx^{\nu_1} \dots dx^{\nu_j}), \quad (2.25)$$

and \star represents the Moyal product $f(x) \star g(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu^x \partial_\nu^y} f(x) g(y) \Big|_{y \rightarrow x}$ with an antisymmetric $\theta^{\mu\nu}$.

$SU(1, 1)$ gauge fields $\hat{A}^{(\pm)a}$ ($a = 0, 1, 2$) that correspond to those of the commutative Chern-Simons theory are combined with two extra $U(1)$ gauge fields \hat{B}_μ to form $U(1, 1)$ gauge fields

$$\mathcal{A}_\mu^{(\pm)A} \tau_A = \hat{A}_\mu^{(\pm)a} \tau_a + \hat{B}_\mu^{(\pm)} \tau_3, \quad (2.26)$$

where $\mathcal{A}^{(\pm)3} = \hat{B}_\mu^{(\pm)}$ and the convention for the generators are summarized in Appendix A.

The equations of motion derived from \hat{I}_{CS} read

$$\begin{aligned} \frac{\delta I_{CS}}{\delta \hat{A}_\mu^{(\pm)a}} &= \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[\eta_{ab} \partial_\rho \hat{A}_\sigma^{(\pm)b} - \frac{\epsilon_{abc}}{2} \hat{A}_\rho^{(\pm)b} \star \hat{A}_\sigma^{(\pm)c} + \frac{i}{6} \eta_{ab} (\hat{A}_\rho^{(\pm)b} \star \hat{B}_\sigma^{(\pm)} + \hat{B}_\rho^{(\pm)} \star \hat{A}_\sigma^{(\pm)b}) \right] \\ &= 0, \end{aligned} \quad (2.27)$$

$$\frac{\delta I_{CS}}{\delta \hat{B}_\mu^{(\pm)}} = \pm \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[\partial_\rho \hat{B}_\sigma^{(\pm)} - \frac{i}{6} \eta_{ab} \hat{A}_\rho^{(\pm)a} \star \hat{A}_\sigma^{(\pm)b} + \frac{i}{2} \hat{B}_\rho^{(\pm)} \star \hat{B}_\sigma^{(\pm)} \right] = 0. \quad (2.28)$$

In the commutative limit $\theta \rightarrow 0$, these equations boil down to the following decoupled equations of motion,

$$F^{(\pm)a} = 0, \quad dB^{(\pm)} = 0. \quad (2.29)$$

2.4 Seiberg-Witten map

The Seiberg-Witten map [17] is introduced as a map between gauge theories on commutative and noncommutative geometries. As shown in [18], Chern-Simons theory has a peculiar feature under the map; the form of the action remains unchanged (up to surface terms), and we can just replace the products with the Moyal products. This property suggests that at least for the part of Chern-Simons action, a solution of the equations of motion can be mapped into that of the noncommutative version of the equations of motion.

We now consider the Seiberg-Witten map based on the radius-angle commutation relation [14]

$$[\hat{R}, \hat{\varphi}] = 2i\theta \quad (2.30)$$

where $\hat{R} = \hat{r}^2$. Namely, $\theta^{R\varphi} = -\theta^{\varphi R} = 2\theta$ and the other components are all zero. The convention is fixed in Appendix A.1, and the correction term from the Seiberg-Witten map is

$$\begin{aligned} A'_\mu(A) = & -\frac{i}{4}(2\theta) \left[\frac{1}{2}\eta_{ab}A_R^a(\partial_\varphi A_\mu^b + F_{\varphi\mu}^b)\mathbf{1} - \frac{1}{2}\eta_{ab}A_\varphi^a(\partial_R A_\mu^b + F_{R\mu}^b)\mathbf{1} \right. \\ & + i(A_R^a\tau_a + B_R\tau_3)(\partial_\varphi B_\mu + F_{\varphi\mu}^{(B)}) - i(A_\varphi^a\tau_a + B_\varphi\tau_3)(\partial_R B_\mu + F_{R\mu}^{(B)}) \\ & \left. + iB_R(\partial_\varphi A_\mu^b + F_{\varphi\mu}^b)\tau_b - iB_\varphi(\partial_R A_\mu^b + F_{R\mu}^b)\tau_b \right]. \end{aligned} \quad (2.31)$$

Since the noncommutative version of Chern-Simons theory has two extra gauge fields $B_\mu^{(\pm)}$, we need to give their forms in the commutative case. In the commutative limit, they have vanishing field strength, $dB^{(\pm)} = 0$. We consider the case with $B_\mu^{(\pm)} = Bd\varphi$ with a constant B . Then the Seiberg-Witten map now reads

$$A_\mu^{(\pm)a'} = -\frac{\theta B}{2}[\partial_R A_\mu^{(\pm)a} + F_{R\mu}^a], \quad (2.32)$$

$$B_\mu^{(\pm)'} = -\frac{\theta}{2}\eta_{ab}[A_R^{(\pm)a}F_{\varphi\mu}^b - A_\varphi^{(\pm)a}F_{R\mu}^b - A_\varphi^{(\pm)a}\partial_R A_\mu^{(\pm)b}]. \quad (2.33)$$

By applying this map to the gauge fields (2.21)–(2.23), to the first order in θ , the noncommutative gauge fields are

$$\hat{A}_t^{(\pm)0} = \pm \frac{1}{\ell} \left(\sqrt{h} \mp h' \frac{\gamma}{2r\sqrt{h}} \right) - \theta B \frac{(2r^2 - \beta)^2 \gamma \pm 2\ell^2(2r^4 - \beta r^2 \mp 4\beta\gamma)h}{16\ell^5 r^4 h^{3/2}}, \quad (2.34)$$

$$\hat{A}_\varphi^{(\pm)0} = - \left(\sqrt{h} \mp \frac{1}{\ell^2} \frac{\gamma}{\sqrt{h}} \right) + \theta B \frac{\pm \gamma(2r^2 - \beta) + 2\ell^2(r^2 - \beta)h}{8\ell^4 r^2 h^{3/2}}, \quad (2.35)$$

$$\begin{aligned} \hat{A}_r^{(\pm)1} = & \frac{1}{\ell\sqrt{f}} \left(\frac{\gamma h'}{2rh(r)} \pm 1 \right) + \frac{\theta B}{32\ell^7 r^6 h^2 f^{3/2}} \left[4\gamma^3(2r^2 - \beta)^2 + 2\ell^2 h [3\gamma r^2(2r^2 - \beta)^2 - 4\beta\gamma^3 \right. \\ & \left. + 2\ell^2 r^2 h (\pm r^2(2r^2 - \beta) + \gamma(2r^2 - 3\beta) \pm 2\ell^2 r^2 h) \right], \end{aligned} \quad (2.36)$$

$$\hat{A}_t^{(\pm)2} = -\frac{h'}{2} \sqrt{\frac{f}{h}} + \theta B \frac{-\gamma^2(2r^2 - \beta)^2 + 2\ell^2 h [4\beta\gamma^2 + 2\ell^2 r^2 h(r^2 + \beta)]}{16\ell^6 r^5 h^{3/2} f^{1/2}}, \quad (2.37)$$

$$\hat{A}_\varphi^{(\pm)2} = \pm \frac{r}{\ell} \sqrt{\frac{f}{h}} \mp \theta B \frac{2\beta\gamma(\pm \ell^2 h + \gamma) + 4r^2(\ell^4 h^2 - \gamma^2)}{16\ell^5 r^3 h^{3/2} f^{1/2}}, \quad (2.38)$$

$$\hat{B}^{(\pm)} = \left(B + \frac{\beta\theta}{4\ell^2 r^2} \right) d\varphi - \beta\theta \frac{\pm r^2 + 2\gamma}{4\ell^3 r^4} dt, \quad (2.39)$$

where the prime ' denotes the r derivative.

In the following section, we discuss black hole solutions in noncommutative gravity based on these expressions. Note that since gauge fields are all functions of r only, we can replace \star product with a usual product.

3 Noncommutative charged rotating BTZ black holes and torsion gravity

3.1 θ -deformed metric

In Section 2, we have derived the noncommutative Chern-Simons gauge fields (2.34)–(2.39). From these, we can reconstruct noncommutative vielbeins and spin connections are

$$\hat{e}^a = \frac{\ell}{2}(\hat{A}^{(+a)} - \hat{A}^{(-a)}), \quad \hat{\omega}^a = \frac{1}{2}(\hat{A}^{(+a)} + \hat{A}^{(-a)}), \quad (3.1)$$

and the explicit forms are

$$\hat{e}^0 = \left(\sqrt{h} - \theta B \frac{2r^2 - \beta}{8\ell^2 r^2 \sqrt{h}} \right) dt + \frac{\gamma}{\ell \sqrt{h}} \left(1 + \theta B \frac{2r^2 - \beta}{8h\ell^2 r^2} \right) d\varphi, \quad (3.2)$$

$$\hat{e}^1 = \left[\frac{1}{\sqrt{f}} + \theta B \frac{2\ell^2 h + 2r^2 - \beta}{8\ell^2 r^2 f^{3/2}} \right] dr, \quad (3.3)$$

$$\hat{e}^2 = \left[r \sqrt{\frac{f}{h}} - \theta B \frac{2\ell^4 r^2 h^2 - (2r^2 - \beta)\gamma^2}{8\ell^4 r^3 h^{3/2} f^{1/2}} \right] d\varphi, \quad (3.4)$$

$$\hat{\omega}^0 = \left[-\frac{\gamma h'}{2\ell r \sqrt{h}} + \theta B \gamma \frac{8\ell^2 \beta h - (2r^2 - \beta)^2}{16\ell^5 r^4 h^{3/2}} \right] dt + \left[-\sqrt{h} + \theta B \frac{r^2 - \beta}{4\ell^2 r^2 \sqrt{h}} \right] d\varphi, \quad (3.5)$$

$$\hat{\omega}^1 = \left[\frac{\gamma h'}{2\ell r h \sqrt{f}} + \theta B \gamma \frac{2\ell^4 r^2 r^2 (2r^2 - 3\beta) + 2\gamma^2 (2r^2 - \beta)^2 + \ell^2 h (12r^6 - 12\beta r^4 + 3\beta^2 r^2 - 4\beta\gamma^2)}{16\ell^7 r^6 h^2 f^{3/2}} \right] dr, \quad (3.6)$$

$$\hat{\omega}^2 = \left[-\frac{h'}{2} \sqrt{\frac{f}{h}} + \theta B \frac{4\ell^4 r^2 h^2 (r^2 + \beta) + 8\beta\gamma^2 \ell^2 h - (2r^2 - \beta)^2 \gamma^2}{16\ell^6 r^5 h^{3/2} f^{1/2}} \right] dt - \theta B \frac{\beta\gamma}{8\ell^3 r^3 \sqrt{f} h} d\varphi. \quad (3.7)$$

From the vielbeins, the deformed metric reads, to the first order in θ , (recall that we can replace the \star product with the usual product since \hat{e}^a are all function of only r)

$$\begin{aligned} ds^2 &= -(\hat{e}^0)^2 + (\hat{e}^1)^2 + (\hat{e}^2)^2 \\ &= - \left[h(r) - \theta B \frac{2r^2 - \beta}{4\ell^2 r^2} \right] dt^2 + \left[\frac{1}{f(r)} + \theta B \frac{2h(r)\ell^2 + 2r^2 - \beta}{4\ell^2 r^2 f(r)^2} \right] dr^2 + \left[r^2 - \frac{\theta B}{2} \right] d\varphi^2 - \frac{2\gamma}{\ell} dt d\varphi \\ &\quad + \mathcal{O}(\theta^2). \end{aligned} \quad (3.8)$$

In the neutral limit $Q \rightarrow 0$ (namely $\beta \rightarrow 0$), this metric agrees with (1.1). When the same change of coordinates as (1.5),

$$r = \tilde{r} - \frac{\theta B}{4\tilde{r}}, \quad (3.9)$$

is applied, one finds that the metric comes back to the undeformed metric (2.15) with r replaced with \tilde{r} . This implies that the Einstein equation

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} - \frac{1}{\ell^2}\hat{g}_{\mu\nu} = 8\pi G\hat{\mathcal{T}}_{\mu\nu}, \quad (3.10)$$

is satisfied if we apply the same coordinate transformation to the right hand side, namely the gauge field energy-momentum tensor.

We look at the change of coordinates and the Einstein equation more closely. The Ricci tensor and the scalar curvature are constructed based on the deformed metric $\hat{g}_{\mu\nu}$ and the Levi-Civita connection,

$$\left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} = \frac{1}{2}\hat{g}^{\rho\sigma}(\partial_\mu\hat{g}_{\nu\sigma} + \partial_\nu\hat{g}_{\mu\sigma} - \partial_\sigma\hat{g}_{\mu\nu}). \quad (3.11)$$

We may write the left hand side of (3.10), the Einstein tensor including the cosmological constant term, as $G_{\mu\nu}^{(\Lambda)}(\hat{g}, \{ \})$. On the other hand, the deformed energy-momentum tensor is

$$\hat{\mathcal{T}}_{\mu\nu} = \frac{1}{\lambda^2} \left[\hat{f}_{\mu\rho}\hat{f}_{\nu\sigma}\hat{g}^{\rho\sigma} - \frac{1}{4}\hat{g}_{\mu\nu}\hat{f}_{\rho\sigma}\hat{f}_{\xi\zeta}\hat{g}^{\rho\xi}\hat{g}^{\sigma\zeta} \right], \quad (3.12)$$

where

$$\hat{f}_{tr}(r) = f_{tr}(r) \Big|_{r \rightarrow r + \frac{\theta B}{4r}} = \frac{\lambda Q}{r} \left(1 - \frac{\theta B}{2r^2} \right) + \mathcal{O}(\theta^2), \quad (3.13)$$

is obtained by applying the inverse of the coordinate transformation (3.9) to the undeformed field strength $f_{tr}(r)$. Then, two equations of motions are related as

$$G_{\mu\nu}^{(\Lambda)}(g, \{ \}) = 8\pi G\mathcal{T}_{\mu\nu} \xrightarrow{r \rightarrow r + \frac{\theta B}{4r}} G_{\mu\nu}^{(\Lambda)}(\hat{g}, \{ \}) = 8\pi G\hat{\mathcal{T}}_{\mu\nu}. \quad (3.14)$$

It therefore seems that the deformed metric is again obtained by a simple coordinate transformation and then is trivial. However, it turns out that there appears a non-vanishing torsion tensor in this charged case and the solution is not related to the undeformed metric just by a coordinate change, as argued in the next subsection.

3.2 Torsion and Einstein-Cartan gravity

The connection is calculated by use of the deformed vielbeins and spin connections (3.2)–(3.7) as

$$\Gamma_{\mu\nu}^\lambda = \hat{e}^\lambda{}_a (\partial_\mu \hat{e}_\lambda{}^a + \hat{\omega}_\mu{}^a{}_b \hat{e}_\lambda{}^b). \quad (3.15)$$

They appear to be not symmetric with respect to μ and ν indices, and then provide nontrivial torsion,

$$T_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho, \quad (3.16)$$

whose non-vanishing components are

$$T_{tr}{}^0 = -\beta\theta B \frac{r^2\ell^2 f + \gamma^2}{8\ell^4 r^5 h^{1/2} f}, \quad T_{r\varphi}{}^0 = -\beta\theta B \frac{\gamma}{8\ell^3 r^3 f h^{1/2}}, \quad (3.17)$$

$$T_{tr}^2 = -\beta\theta B \frac{\gamma}{4\ell^3 r^4 (fh)^{1/2}}, \quad T_{r\varphi}^2 = -\beta\theta B \frac{1}{8\ell^2 r^2 (fh)^{1/2}}. \quad (3.18)$$

Therefore, the connection associated with the deformed solution is not a Levi-Civita connection but a more general Affine connection. The curvature tensors should then be calculated by use of $\hat{g}_{\mu\nu}$ and $\Gamma_{\mu\nu}^\rho$, and the non-vanishing components of Einstein tensor including a cosmological constant term $G_{\mu\nu}^{(\Lambda)}(\hat{g}, \Gamma)$ are

$$G_{tt}^{(\Lambda)}(\hat{g}, \Gamma) = \beta \frac{\ell^2 r^2 h + 2\gamma^2}{2\ell^4 r^4} + \beta\theta B \frac{8\ell^2 r^2 f - 2r^4 + \beta r^2 + 24\gamma^2}{16\ell^4 r^6}, \quad (3.19)$$

$$G_{t\varphi}^{(\Lambda)}(\hat{g}, \Gamma) = -\beta \frac{\gamma}{2\ell^3 r^2} - \beta\gamma\theta B \frac{3}{4\ell^3 r^4}, \quad (3.20)$$

$$G_{rr}^{(\Lambda)}(\hat{g}, \Gamma) = -\beta \frac{1}{2\ell^2 r^2 f} - \beta\theta B \frac{10\ell^2 r^2 f + 2r^4 - \beta r^2 - 2\gamma^2}{16\ell^4 r^6 f^2}, \quad (3.21)$$

$$G_{\varphi t}^{(\Lambda)}(\hat{g}, \Gamma) = -\beta \frac{\gamma}{2\ell^3 r^2} - \beta\gamma\theta B \frac{1}{\ell^3 r^4}, \quad (3.22)$$

$$G_{\varphi\varphi}^{(\Lambda)}(\hat{g}, \Gamma) = \beta \frac{1}{2\ell^2} + \beta\theta B \frac{3}{8\ell^2 r^2}. \quad (3.23)$$

Note that the Einstein tensor $G_{\mu\nu}^{(\Lambda)}(\hat{g}, \Gamma)$ is not symmetric due to torsion. Since torsion transforms as genuine tensor, this solution cannot be related to the undeformed solution with vanishing torsion by a mere coordinate change.

The theory of gravity with torsion is known as Einstein-Cartan theory of gravitation. Some features of Einstein-Cartan theory are briefly summarized in Appendix A.3. As explained there, the equations of motion has an extra contribution depending on torsion, and they now read

$$G_{\mu\nu}^{(\Lambda)}(\hat{g}, \Gamma) - \frac{1}{2} \nabla_\alpha^* [-\tilde{T}_{\mu\nu}^\alpha + \tilde{T}^\alpha_{\mu\nu} + \tilde{T}^\alpha_{\nu\mu}] = 8\pi G \tilde{T}_{\mu\nu}, \quad (3.24)$$

where $\nabla_\alpha^* = \nabla_\alpha + T_\alpha$, T_α the trace of the torsion tensor, and $\tilde{T}_{\mu\nu}^\rho$ the deformed torsion tensor. Note that (3.18) leads to the vanishing trace of the torsion $T_\alpha = 0$ and then $\nabla_\alpha^* = \nabla_\alpha$.

Now we make an interesting observation; the equations of motion (3.24) are also satisfied if the deformed energy momentum tensor (3.12) is adopted. Namely, we have confirmed the following equivalence,

$$\begin{aligned} G_{\mu\nu}^{(\Lambda)}(g, \{ \}) &\xrightarrow{r \rightarrow r + \frac{\theta B}{4r}} G_{\mu\nu}^{(\Lambda)}(\hat{g}, \{ \}) \\ &= G_{\mu\nu}^{(\Lambda)}(\hat{g}, \Gamma) - \frac{1}{2} \nabla_\alpha^* [-\tilde{T}_{\mu\nu}^\alpha + \tilde{T}^\alpha_{\mu\nu} + \tilde{T}^\alpha_{\nu\mu}]. \end{aligned} \quad (3.25)$$

Putting it in another way; in the left hand side of (3.24), the effects of torsion appear to cancel out.

So far we have observed a part of the set of equations of motion. In Einstein-Cartan theory, there are equations of motion for the variation with respect to the torsion,

$$K_\rho^{\nu\mu} + T^\nu \delta_\rho^\mu - T^\mu \delta_\rho^\nu = -\frac{\delta I_{\text{gauge}}}{\delta T_{\mu\nu}^\rho}, \quad (3.26)$$

where $K_\rho^{\nu\mu}$ is the contortion. If we consider the standard $U(1)$ gauge field action on the deformed solution, the right hand side is zero, as the action does not couple to torsion. Since the left hand side is non-zero, we need to modify the matter part of the action. These two equations of motions are treated in a unified way by use of Chern-Simons equations of motion. In the next subsection, we come back to Chern-Simons equations.

3.3 The Chern-Simons equations of motion and the matter energy-momentum tensor

After the deformation, we may assume that the matter part action is replaced as

$$\hat{I}_{\text{gauge}} = -\frac{1}{\lambda^2} \int d^3x \sqrt{-\hat{g}} f_{\mu\nu} f_{\rho\sigma} \hat{g}^{\mu\rho} \hat{g}^{\nu\sigma} + I'_{\text{gauge}}, \quad (3.27)$$

where I'_{gauge} would depend on the spin connection (or torsion), and will vanish in the commutative limit $\theta \rightarrow 0$. The equations of motion are

$$\begin{aligned} \frac{\delta I_{CS}}{\delta \hat{A}_\mu^{(\pm)a}} &= \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[\eta_{ab} \partial_\rho \hat{A}_\sigma^{(\pm)b} - \frac{\epsilon_{abc}}{2} \hat{A}_\rho^{(\pm)b} \star \hat{A}_\sigma^{(\pm)c} + \frac{i}{6} \eta_{ab} (\hat{A}_\rho^{(\pm)b} \star \hat{B}_\sigma^{(\pm)} + \hat{B}_\rho^{(\pm)} \star \hat{A}_\sigma^{(\pm)b}) \right] \\ &= -\frac{\delta \hat{I}_{\text{gauge}}}{\delta \hat{A}_\mu^{(\pm)a}}, \end{aligned} \quad (3.28)$$

$$\frac{\delta I_{CS}}{\delta \hat{B}_\mu^{(\pm)}} = \pm \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \left[\partial_\rho \hat{B}_\sigma^{(\pm)} - \frac{i}{6} \eta_{ab} \hat{A}_\rho^{(\pm)a} \star \hat{A}_\sigma^{(\pm)b} + \frac{i}{2} \hat{B}_\rho^{(\pm)} \star \hat{B}_\sigma^{(\pm)} \right] = -\frac{\delta \hat{I}_{\text{gauge}}}{\delta \hat{B}_\mu^{(\pm)}}, \quad (3.29)$$

$$\frac{\delta \hat{I}_{\text{gauge}}}{\delta a_\mu} = \frac{1}{\lambda^2} \nabla_\rho f^{\rho\mu} + \frac{\delta I'_{\text{gauge}}}{\delta a_\mu} = 0. \quad (3.30)$$

We require that the last equation (3.30) leads to the solution $\tilde{f}_{\mu\nu}$ in (3.13). We also consider the case in which all the fields are functions of only r . Then, we can replace \star product with a usual product, and the interactions between $\hat{A}_\mu^{(\pm)a}$ and $\hat{B}_\mu^{(\pm)}$ are dropped. The left hand side of (3.28) is

$$\begin{aligned} \frac{\delta \hat{I}_{CS}}{\delta \hat{A}_\mu^{(\pm)a}} &= \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \eta_{ab} \left[\partial_\rho \hat{A}_\sigma^{(\pm)b} - \frac{\epsilon^b_{cd}}{2} \hat{A}_\rho^{(\pm)c} \hat{A}_\sigma^{(\pm)d} \right] \\ &= \mp \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \eta_{ab} \left[\partial_\rho \hat{\omega}_\sigma^b - \frac{\epsilon^b_{cd}}{2} \left(\hat{\omega}_\rho^c \hat{\omega}_\sigma^d + \frac{1}{\ell^2} \hat{e}_\rho^c \hat{e}_\sigma^d \right) \pm \frac{1}{\ell} (\partial_\rho \hat{e}_\sigma^b - \epsilon^b_{cd} \hat{e}_\rho^c \hat{\omega}_\sigma^d) \right] \\ &= \pm \frac{k}{4\pi} g^{\mu\zeta} e^\delta_a \sqrt{-g} G_{\zeta\delta}^{(\Lambda)}(\hat{g}, \Gamma) - \frac{k}{8\pi\ell} \epsilon^{\mu\rho\sigma} \eta_{ab} T_{\rho\sigma}{}^b. \end{aligned} \quad (3.31)$$

In the left hand side of (3.29), it is easy to see that only the first term survives. Thus, the equations of motion are

$$\pm \frac{k}{4\pi} \hat{g}^{\mu\zeta} \hat{e}^\delta_a \sqrt{-\hat{g}} G_{\zeta\delta}^{(\Lambda)}(\hat{g}, \Gamma) - \frac{k}{8\pi\ell} \epsilon^{\mu\rho\sigma} \eta_{ab} T_{\rho\sigma}{}^b = \mp \frac{\ell}{2} \eta_{ac} \cdot \sqrt{-\hat{g}} \hat{T}^{\mu\rho} \hat{e}_\rho^c - \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\mu^{(\pm)a}}, \quad (3.32)$$

$$\pm \frac{k}{4\pi} \epsilon^{\mu\rho\sigma} \partial_\rho \hat{B}_\sigma^{(\pm)} = -\frac{\delta I'_{\text{gauge}}}{\delta \hat{B}_\mu^{(\pm)}}. \quad (3.33)$$

By rearranging them, we find

$$G_{\mu\nu}^{(\Lambda)}(\hat{g}, \Gamma) = 8\pi G \hat{T}_{\mu\nu} - \frac{2\pi}{k} \hat{g}_{\mu\zeta} \hat{e}_\nu^a \left(\frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\zeta^{(+)a}} - \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\zeta^{(-)a}} \right), \quad (3.34)$$

$$T_{\mu\nu}{}^a = 8\pi G \eta^{ab} \epsilon_{\mu\nu\zeta} \left(\frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\zeta^{(+)b}} + \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\zeta^{(-)b}} \right). \quad (3.35)$$

The first equation implies that

$$\frac{2\pi}{k} \hat{g}_{\mu\zeta} \hat{e}_\nu{}^a \left(\frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\zeta^{(+a)}} - \frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\zeta^{(-a)}} \right) = -\frac{1}{2} \nabla_\alpha^* [-\tilde{T}_{\mu\nu}{}^\alpha + \tilde{T}^\alpha{}_{\mu\nu} + \tilde{T}^\alpha{}_{\nu\mu}]. \quad (3.36)$$

We therefore find the set of conditions for the correction term for the matter part as

$$\frac{\delta I'_{\text{gauge}}}{\delta \hat{A}_\mu^{(\pm)a}} = \pm \frac{\ell}{16\pi G} \hat{g}^{\mu\nu} \hat{e}_\nu{}^\rho \nabla_\alpha^* [-\tilde{T}_{\nu\rho}{}^\alpha + \tilde{T}^\alpha{}_{\nu\rho} + \tilde{T}^\alpha{}_{\rho\nu}] - \frac{1}{32\pi G} \epsilon^{\mu\nu\rho} \eta_{ab} T_{\nu\rho}{}^b, \quad (3.37)$$

$$\frac{\delta I'_{\text{gauge}}}{\delta \hat{B}_\mu^{(\pm)}} = \pm \frac{\ell}{16\pi G} \epsilon^{\mu\rho\sigma} \partial_\rho \hat{B}_\sigma^{(\pm)}, \quad (3.38)$$

$$\frac{\delta I'_{\text{gauge}}}{\delta a_\mu} = -\frac{1}{\lambda^2} \nabla_\rho \hat{f}^{\rho\mu}. \quad (3.39)$$

We have not fixed the explicit form of the correction term yet. So we just present the necessary conditions for the θ dependent correction term for the matter part of the action.⁴

3.4 Coordinate change

Finally, we briefly comment on the change of the coordinate (1.5). There is a subtle point on the regions that the radial coordinate covers. In (3.9), r is a monotonically increasing function of \tilde{r} , and $r \geq 0$ corresponds to $\tilde{r} \geq \sqrt{\theta B}/2$. In (3.8) (now we use \tilde{r} for the deformed metric), the radial part $(\tilde{r}^2 - \frac{\theta B}{2})d\varphi^2$ suggests that only the part of the geometry with $\tilde{r} \geq \sqrt{\theta B}/\sqrt{2}$ is meaningful. This part corresponds to $r \geq \frac{\sqrt{\theta B}}{2\sqrt{2}}$. Therefore, the undeformed geometry except the vicinity of the center, $0 \leq r \leq \frac{\sqrt{\theta B}}{2\sqrt{2}}$, is mapped to the whole region of the deformed geometry by (3.9). Let r_+ be the radius of the outer horizon of the undeformed BTZ solution (2.16); namely the largest root of $f(r) = 0$. The r coordinate covers $r \geq r_+$. Therefore, if $r_+ > \sqrt{\theta B}/2\sqrt{2}$, the whole region outside the horizon of the undeformed solution is mapped. On the other hand, if $r_+ < \sqrt{\theta B}/2\sqrt{2}$, the part $r_+ \leq r \leq \sqrt{\theta B}/2\sqrt{2}$ will be missing in the deformed solution. This may not be so surprising; in the current formulation, the noncommutative parameter appears only in the combination of θB and $\sqrt{\theta B}$ serves a characteristic length scale in the noncommutative geometry. It is in a sense a minimum length and a black hole of the size $r_+ < \sqrt{\theta B}/2\sqrt{2}$ is not well-defined in the noncommutative side.

4 Conclusion

In this paper, we have explored charged rotating BTZ black hole geometries by use of Chern-Simons formulation of 2 + 1 dimensional gravity and the Seiberg-Witten map.

The noncommutativity in question is the one between the radial coordinate and the angular coordinate as $[r^2, \varphi] = 2i\theta$. The noncommutative deformation for the pure gravity part is introduced by the Seiberg-Witten map for the Chern-Simons gauge fields where two extra $U(1)$ gauge fields are added when noncommutative deformation is considered. The deformation of the matter gauge field part is to be determined so that the equations of motion are satisfied.

⁴By use of the change of the variables (A.11), one can also consider these relations in terms of the variations with respect to the metric and the torsion.

It is found that as with the neutral case, the deformed metric is related to the undeformed one via a simple coordinate change. Through this observation, we discuss that the deformation of the matter energy-momentum tensor can also be obtained by the same coordinate change. However, there appears torsion that is proportional to the parameter of noncommutativity and it cannot be transformed away. We thus analyze the equations of motion with the framework of Einstein-Cartan torsion gravity. It is found that with the same deformed matter energy-momentum tensor the equations of motion of the torsion gravity are satisfied. Namely, the effect of the torsion is somehow compensated in the pure gravity part, and does not affect the matter part. We have not yet fixed the deformed matter part action completely, but we have presented a set of equations for the correction term to satisfy.

There are several issues that are still unclear. First thing is that why the noncommutative deformation is represented by a simple change of the radial coordinate. One can check that the result of the Seiberg-Witten map for the difference of the gauge fields, $A_\mu^{(+a)} - A_\mu^{(-a)}$, can be obtained by the coordinate change, but the sum is not. Therefore, the deformed vielbein \hat{e}_μ^a are related to the undeformed one via the coordinate change, but the spin connections are not. This difference leads to nontrivial torsion of the deformed background. The gauge field representation has some gauge degrees of freedom, but the appearance of the torsion is physical effect and it does not seem to be trivially eliminated by the coordinate change.

Secondly, the admitted minimal black holes discussed in the section 3.4 may imply that the noncommutative space-time has its own entropy, i.e. $S \propto \theta B$ in a region of Planckian size, presuming the area law still applies. This reminds us of the spin foam model in the loop quantum gravity [19] and we wonder if e^S counts the spin combination.

Thirdly, it is curious how the properties of the charged BTZ black hole are changed (or not changed). The torsion may affect the property of black holes through the change of the metric [20]. However, in our case as long as we look at the metric only, we do not see the difference. It is interesting to see if this is a peculiar feature of the current solution, or this may happen in a broader setup of $2 + 1$ dimensional gravity with noncommutativity. On top of that, it should also be important to fix the deformation of the matter part action and examine how the matter part action couples to the torsion or the extra $U(1)$ gauge fields $B_\mu^{(\pm)}$.

Finally, we mention a discussion made in [8]. There Lie algebraic types of general noncommutative structures are considered in $3 + 1$ dimensional gravity, and it is argued that the first order corrections vanish under the condition of vanishing classical torsion. In our construction, we include a matter field whose deformation is not completely fixed by the Seiberg-Witten map and the deformed solution has nontrivial torsion. Thus our result would not be immediately contradict to the result of [8]. Since there appears a simple relation between the deformed geometry with torsion and the undeformed geometry, it may be interesting to investigate the extension of the argument for torsion gravity.

Note added: In completing this work, there appeared a paper [21] which considered a non-commutative deformation in four dimensional gravity. They also observed the emergence of torsion.

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A Conventions and notations

We summarize our conventions and notations in this paper here.

A.1 Seiberg-Witten map

Seiberg and Witten showed that a field theory on D-branes with a background B field can be formulated as a conventional Yang-Mills theory or noncommutative Yang-Mills theory depending on the regulator we choose, Pauli-Villars or point-splitting respectively [17]. The gauge transformation is now defined by use of Moyal product as

$$\begin{aligned}\hat{\delta}_{\hat{\xi}}\hat{A}_{\mu} &= \partial_{\mu}\hat{\xi} - \hat{\xi} \star \tilde{A}_{\mu} + \hat{A}_{\mu} \star \hat{\xi} \\ &= \partial_{\mu}\hat{\xi} - \frac{i}{2}\theta^{\nu\rho}(\partial_{\nu}\hat{\xi}\partial_{\rho}\hat{A}_{\mu} - \partial_{\nu}\hat{A}_{\mu}\partial_{\rho}\hat{\xi}) + \mathcal{O}(\theta^2).\end{aligned}\tag{A.1}$$

The Seiberg-Witten map is defined as a compatibility condition of gauge transformation and a mapping between A and \hat{A} ,

$$\hat{A}(A) + \hat{\delta}_{\hat{\xi}}\hat{A}(A) = \hat{A}(A + \delta_{\xi}A),\tag{A.2}$$

for infinitesimal ξ and $\hat{\xi}$. The solution is

$$\hat{A}_{\mu}(A) = A_{\mu} - \frac{i}{4}\theta^{\nu\rho}\{A_{\nu}, \partial_{\rho}A_{\mu} + F_{\rho\mu}\} + \mathcal{O}(\theta^2),\tag{A.3}$$

$$\hat{\xi}(\xi, A) = \xi + \frac{i}{4}\theta^{\mu\nu}\{\partial_{\mu}\xi, A_{\nu}\} + \mathcal{O}(\theta^2),\tag{A.4}$$

where $\{f, g\} = fg + gf$ is the anti-commutator with the conventional matrix product.

A.2 Some notations and $U(1, 1)$ generators

The epsilon tensor is $\epsilon_{012} = -\epsilon^{012} = 1$. We define for a spin connection 1-form ω^a_b ,

$$\omega_a = -\frac{1}{2}\epsilon_{abc}\omega^{bc}.\tag{A.5}$$

For the neutral BTZ black holes, r_{\pm} is defined by

$$r_{\pm}^2 = 4G\ell^2\left(M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}}\right),\tag{A.6}$$

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad J = \frac{r_+r_-}{4G\ell}.\tag{A.7}$$

Our convention of $U(1, 1)$ generators is

$$\tau_0 = \frac{i}{2}\sigma_3, \quad \tau_1 = \frac{1}{2}\sigma_1, \quad \tau_2 = \frac{1}{2}\sigma_2, \quad \tau_3 = \frac{i}{2}\mathbf{1}_2,\tag{A.8}$$

with $a, b = 0, 1, 2$, $A, B = 0, 1, 2, 3$, $\eta_{AB} = \text{diag}(-1, 1, 1, -1)$ and they satisfy

$$g_{AB} = \text{tr}(\tau_A \tau_B) = \frac{1}{2} \eta_{AB}, \quad [\tau_A, \tau_B] = -\epsilon_{AB}^C \tau_C, \quad \epsilon_{AB}^C = \begin{cases} \epsilon_{ab}^c \\ \epsilon_{ab}^3 = \epsilon_{3a}^b = 0 \end{cases}, \quad (\text{A.9})$$

$$\{\tau_a, \tau_b\} = \frac{1}{2} \eta_{ab} \mathbf{1}, \quad \{\tau_A, \tau_3\} = i\tau_A, \quad \text{tr}(\tau_a \tau_b \tau_c) = -\frac{1}{4} \epsilon_{abc}, \quad \text{tr}(\tau_a \tau_b \tau_3) = \frac{i}{4} \eta_{ab}. \quad (\text{A.10})$$

By use of the chain rule, we can convert the variation with respect to the gauge fields to those with respect to the metric and the torsion as

$$\begin{aligned} \frac{\delta}{\delta A_{\mu}^{(\pm)a}} = & \mp \frac{\ell}{2} \left[2g^{\mu\alpha} e^\beta_a \frac{\delta}{\delta g^{\alpha\beta}} + e^\beta_a [\delta_\beta^\alpha T_{\rho\sigma}{}^\mu + \delta_\rho^\mu \Gamma_{\sigma\beta}^\alpha - \delta_\sigma^\mu \Gamma_{\rho\beta}^\alpha] \frac{\delta}{\delta T_{\rho\sigma}{}^\alpha} \right] \\ & + \frac{1}{2} \epsilon_{ca}^b e^\alpha_b (\delta_\rho^\mu e_\sigma{}^c - \delta_\sigma^\mu e_\rho{}^c) \frac{\delta}{\delta T_{\rho\sigma}{}^\alpha}. \end{aligned} \quad (\text{A.11})$$

A.3 Einstein-Cartan theory of torsion gravity

The Einstein-Cartan theory of gravitation is a generalization of Einstein's theory of general relativity to allow torsion in space-time. It can be regarded as a gauge theory of the Poincaré symmetry instead of the Lorentz symmetry[23]. While curvature is related to the energy momentum tensor with Lorentz symmetry, torsion is related to the density of intrinsic angular momentum or spin. For some overview of torsion gravity, see [24].

The vielbein e^a gives the metric by $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, and its inverse is $e_\mu{}^a e^\nu{}_a = \delta_\mu^\nu$ and $e_\mu{}^a e^\mu{}_b = \delta_b^a$. With spin connections $\omega_\mu{}^a{}_b$, Affine connections are defined by

$$\Gamma_{\mu\lambda}^\nu = e^\nu{}_a (\partial_\mu e_\lambda{}^a + \omega_\mu{}^a{}_b e_\lambda{}^b), \quad (\text{A.12})$$

and the torsion tensor is

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \hat{\omega}_\mu{}^a{}_b e_\nu{}^b - \hat{\omega}_\nu{}^a{}_b e_\mu{}^b, \quad (\text{A.13})$$

and $T_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho = T_{\mu\nu}{}^a e^\rho{}_a$. The curvature tensor is

$$R^\lambda{}_{\rho\mu\nu} = \partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\nu \Gamma_{\mu\rho}^\lambda + \Gamma_{\mu\xi}^\lambda \Gamma_{\nu\rho}^\xi - \Gamma_{\nu\xi}^\lambda \Gamma_{\mu\rho}^\xi, \quad (\text{A.14})$$

and the Ricci tensor and the scalar curvature are defined by $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$ respectively.

In Einstein-Cartan theory of torsion gravity, the metric $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^\rho$ are treated as independent variables. When we consider the equations of motion, we can take the variation of torsion tensor instead of the connection. The action is given by the usual Einstein-Hilbert form,

$$I_G = \frac{1}{16\pi G} \int d^3x \sqrt{-g} R, \quad (\text{A.15})$$

and the variations are

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta I_G}{\delta g^{\mu\nu}} = G_{\mu\nu}^{(\Lambda)}(g, \Gamma) - \frac{1}{2} \nabla_\alpha^* [-\tilde{T}_{\mu\nu}{}^\alpha + \tilde{T}^\alpha{}_{\mu\nu} + \tilde{T}^\alpha{}_{\nu\mu}], \quad (\text{A.16})$$

$$16\pi G \frac{\delta I_G}{\delta T_{\mu\nu}{}^\rho} = K_\rho{}^{\nu\mu} + T^\nu \delta_\rho^\mu - T^\mu \delta_\rho^\nu, \quad (\text{A.17})$$

where $\overset{*}{\nabla}_\alpha = \nabla_\alpha + T_\alpha$ with ∇_α being a covariant derivative with the connection Γ and T_α being the trace of the torsion tensor $T_{\alpha\nu}{}^a \hat{e}^\nu{}_a$. $K_{\mu\nu\sigma}$ is the contortion tensor

$$K_{\mu\nu\sigma} = \frac{1}{2}(T_{\mu\nu\sigma} - T_{\nu\sigma\mu} + T_{\sigma\mu\nu}), \quad (\text{A.18})$$

and $\tilde{T}_{\mu\nu}{}^\rho$ is known as the deformed torsion tensor and is defined as

$$\tilde{T}_{\mu\nu}{}^\rho = T_{\mu\nu}{}^\rho + \delta_\mu^\rho T_\nu - \delta_\nu^\rho T_\mu. \quad (\text{A.19})$$

Finally, by use of the one forms $e^a = e_\mu{}^a dx^\mu$ and $\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu$, the torsion and the curvature two forms are written as

$$T^a = De^a = de^a + \omega^a{}_b \wedge e^b, \quad (\text{A.20})$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \quad (\text{A.21})$$

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